THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050A Mathematical Analysis I (Fall 2022)
Suggested Solution of Homework 4

(1) (a) Fix $\epsilon > 0$. Since $\sum_{i=1}^{\infty} a_i$ converges, there exists $N \in \mathbb{N}$ such that for any $m, n > N$, $\sum_{i=m}^{n} a_i < \epsilon^{\frac{1}{3}}$. Then $\sum_{i=m}^{n} a_i^3 < (\sum_{i=m}^{n} a_i)^3 < \epsilon$. Hence, $\sum_{i=1}^{\infty} a_i^3$ converges.

(b) Counter-example: $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. (See Textbook Example 3.7.6 for a proof of the divergence.)

(c) Since $a_n > 0$, $b_n > \frac{a_1}{n}$. By Comparison Test, the divergence of $\sum_{i=n}^{\infty} b_n$ follows from that of $\sum_{n=1}^{\infty} \frac{1}{n}$.

(2) (a) Fix $\epsilon > 0$. Take $\delta = \min\{\frac{12\epsilon}{29}, 1\}$.

For any $\epsilon \in (1-\delta, 1+\delta)$, $|\frac{x^3-2}{3+x} + \frac{1}{4}| = |x-1||\frac{4x^2+4x+5}{4x+12}| < \frac{12\epsilon}{29}\frac{29}{12} = \epsilon$.

Hence, $\lim_{x\to 1} \frac{x^3-2}{3+x} = -\frac{1}{4}$.

(b) Fix $\epsilon > 0$. Take $\delta = \epsilon^4$.

For any $\epsilon \in (0, \delta)$, $|x^{\frac{1}{4}} \cos(e^{\frac{1}{x}})| \le |x^{\frac{1}{4}}| < (\epsilon^4)^{\frac{1}{4}} = \epsilon$.

Hence, $\lim_{x\to 0^+} x^{\frac{1}{4}} \cos(e^{\frac{1}{x}}) = 0$.

(3) Take $x_n = \frac{1}{n} + 1$. Then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n} + 1 = 1$. For any $\alpha > 0$, by Archimedean property, there exists $n \in \mathbb{N}$ such that $n > \alpha$. $\exp \frac{1}{\sqrt{x_n}-1} > \exp \frac{1}{x_n - 1} = e^n > n > \alpha$.

Hence, $\lim_{x\to 1} \exp \frac{1}{\sqrt{x}-1}$ does not exist.

(4) See Test 2 Solution Question 4.